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‘Morphing’ bistable orthotropic elliptical shallow shells

By K. A. Seffen*

Department of Engineering, University of Cambridge, Trumpington Street, Cambridge CB2 1PZ, UK

This study is concerned with the equilibrium shapes of orthotropic, elliptical plates and shells deforming elastically without initial stresses. The aim is to explore potential bistable configurations and their dependencies on material parameters and initial shape for elucidating novel morphing structures. A strain energy formulation gives way to a compact set of governing equations of deformation, which can be solved in closed form for some isotropic and orthotropic conditions. It is shown that bistability depends on the change in Gaussian curvature of the shell, in particular, for initially untwisted shells, isotropy precludes bistability, where there is negative initial Gaussian curvature, but orthotropic materials yield bistability irrespective of the sign of the initial Gaussian curvature. This improved range of performance stems from increasing the independent shear modulus, which imparts sufficient torsional rigidity to stabilize against perturbations in the deformed state. It is also shown that the range of bistable configurations for initially twisted shells generally diminishes as the degree of twist increases.

Keywords: bistable; orthotropic; elliptical; morphing structures; Gaussian curvature

1. Introduction

Compared to conventional engineering structures, where sizeable deflexions portend failure, metamorphosing or ‘morphing’ structures are designed to be reconfigured elastically between stable and radically different states. They are being proposed for novel, multi-functional systems, where appreciable departures from the initial shape offer competitive advantages elsewhere. For example, fervent research efforts are focusing on new aircraft concepts in which controlled distortions of key aerodynamical surfaces tailor the flight envelope. Weiss (2003) describes shape-shifting trailing edges that constantly re-adjust the cross-sectional profile of wing for an optimal lift-to-drag ratio, reconfigurable engine nacelles for better thermodynamical efficiency and flight range, and compliant wings for exacerbating roll under torsion for greater agility during dog-fighting.

The design of morphing structures faces clear challenges in delivering the required shape changes while maintaining strength, stiffness and stability. A viable class of simple but effective structures seems to be thin-walled, bistable shells with two separately curved configurations. This performance is wrought by the aggregation and relief of stored strain energy under large deflexions, and for

*kas14@cam.ac.uk

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Table 1. Nomenclature.

a, b	major and minor semi axes lengths of ellipse
E_x, E_y, G	Young's moduli and shear modulus
k_1, k_2, k_3, k_4	arbitrary constants of curvature
M_x, M_y, M_{xy}	bending moments and twisting moment
t	plate thickness
U_B, U_S	strain energy densities of bending and stretching. Dimensionless $\bar{U}=U\cdot 12(1-\nu^2/\beta)a^3/E\pi t^5 b$
x, y, z	orthogonal coordinates within plate; z is normal to the middle surface
α	dimensionless torsional rigidity equal to $2\rho(1-\nu^2/\beta)$
β, ρ	modular ratios, equal to E_y/E_x and G/E_x
Δg	change in Gaussian curvature, $\kappa_x\kappa_y-\kappa_{xy}^2-\kappa_{x0}\kappa_{y0}+\kappa_{xy0}^2$
ϵ_x, ϵ_y, γ_{xy}	direct strains and shear strain
χ_x, χ_y, χ_{xy}	changes in curvatures and twisting curvature, or 'twist'
κ_x, κ_y, κ_{xy}	absolute curvatures. An extra subscript, '0', denotes initial value. Dimensionless values are $\bar{\kappa}=\kappa a^2/t$
σ_x, σ_y, τ_{xy}	direct stresses and shear stress
ν_{yx}, ν_{xy}	orthogonal Poisson's ratios

continuous plates and shells without initial stresses, it can only be governed by the initial shape of structure and the constitutive material behaviour, but in a mutually inclusive way, e.g. bistable isotropic shells are doubly curved (Mansfield 1965), whereas simpler cylindrically curved strips are bistable when made from anisotropic materials such as composites (Guest & Pellegrino 2006). More specifically, material anisotropy distorts the symmetry of stored energy under uniform deformation defined by a particular direction within the shell, thereby seeding preferential directions for minimal stored energy. The initial shape determines whether there is a change in the Gaussian curvature (Calladine 1983), which further develops the interaction between bending and stretching in the nonlinear regime. To date, the combination of the influences of material and shape has not been explored in a systematic way, for it may reveal new bistable forms for augmenting the design and capabilities of novel morphing structures, which is the subject of this paper.

For solutions to become tractable within a large deflexion formulation, two compromises are introduced: the material behaviour is chosen to be *orthotropic* and the initial shape is *uniformly curved*. When mechanical loads are absent, the second compromise obviates a similar uniformity of deformed curvatures except for regions close to any free edges, where a boundary layer of non-uniform deformation must develop (Galletly & Guest 2004). The width of this layer diminishes when the thickness variation near the free edge vanishes. Although no such condition is prescribed here, where a constant thickness greatly simplifies the calculation detail, the shell is assumed to be very thin, so that the boundary-layer width and hence its effect diminishes in comparison to the bulk of the shell, and its contribution to the formulation can be discounted without detriment to the general outcome. The outline of this paper is described subsequently.

Section 2 obtains the large deflexion behaviour of a shell with an elliptical planform, which is chosen for its mathematical expediency and describing either strips or discs by varying its semi-axes. Traditional formulations simultaneously solve a pair of nonlinear governing partial differential equations (Mansfield 1989; Weaver 2006) derived from general equilibrium and compatibility considerations subject to boundary conditions; but the assumption of uniform curvatures *a priori* permits a simpler scheme, as the change in Gaussian curvature is known, which prescribes the variation of in-plane strains and stresses. The total strain energy stored in the shell can then be determined straightforwardly, whose stationary values yield equilibrium configurations, and further calculus procedures help to ascribe the character of their stability. Note that the properties of the equilibria do not depend on any enabling agency of deformation under conservative behaviour and the formulation discounts any external effort, although it would be needed to elucidate the corresponding equilibrium path, if required.

Section 3 first determines the influence of the initial shape under isotropic behaviour. It repeats some of the work of Mansfield (1965), but it introduces an important finding, namely, the influence of the change in Gaussian curvature upon the stability of solution. In addition, the relatively simpler governing equations reveal key solutions in closed form, and this procedure is repeated for a special orthotropic case, where the bending response is isotropic to highlight the stabilizing effects of an increased torsional rigidity under a larger, independent shear modulus. The section then considers two numerical examples, where feasible bistable geometries are established for a practical composite material and a hypothetical material, where there is a constraint in the initial shape. General conclusions are drawn together in §4. For a detailed list of all nomenclature see table 1.

2. Constitutive behaviour and formulation

The symbols for direct stress and strain are σ and ϵ, and a single subscript denotes a coordinate direction. For orthotropic material behaviour, the constitutive relationships for plane stress behaviour can be written as

$$\epsilon_x = \frac{\sigma_x}{E_x} - \frac{\nu_{yx}\sigma_y}{E_y}; \quad \epsilon_y = \frac{\sigma_y}{E_y} - \frac{\nu_{xy}\sigma_x}{E_x}; \quad \gamma_{xy} = \frac{\tau_{xy}}{G}, \tag{2.1}$$

where E_x and E_y are the Young's moduli in the directions of orthogonal coordinates, x and y, respectively, and G is an independent shear modulus relating the shear stress, τ_{xy}, to the shear strain, γ_{xy}. The reciprocal theorem ties together the orthogonal Poisson's ratios, ν_{yx} and ν_{xy}, such that

$$\frac{\nu_{xy}}{E_x} = \frac{\nu_{yx}}{E_y}. \tag{2.2}$$

For straightforwardness, new relationships are defined in terms of constants and Greek symbols,

$$E_x = E, \quad E_y = \beta E_x, \quad G = \rho E_x, \quad \nu_{yx} = \nu, \quad \nu_{xy} = \frac{\nu}{\beta}, \tag{2.3}$$

where for isotropy, $\beta = 1$ and $\rho = 1/2(1+\nu)$ for a homogenous material. The original strains are written as

$$\epsilon_x = \frac{\sigma_x}{E} - \frac{\nu\sigma_y}{\beta E}; \quad \epsilon_y = -\frac{\nu\sigma_x}{\beta E} + \frac{\sigma_y}{\beta E}; \quad \gamma_{xy} = \frac{\tau_{xy}}{\rho E}, \tag{2.4}$$

or, if the expressions are solved explicitly for stresses,

$$\sigma_x = \frac{E}{(1-\nu^2/\beta)}[\epsilon_x + \nu\epsilon_y]; \quad \sigma_y = \frac{E\beta}{(1-\nu^2/\beta)}\left[\epsilon_y + \frac{\nu\epsilon_x}{\beta}\right]; \quad \tau_{xy} = E\rho\gamma_{xy}. \tag{2.5}$$

The governing equations of deformation for large deflexion orthotropic behaviour are derived using a strain energy formulation based on familiar but general components of bending and stretching strain energy densities per unit surface area of plate. Specifically, the assumption of uniform curving defines the changes in curvature and hence the change in Gaussian curvature. The in-plane middle surface strains (and stresses) can be determined from the requirement of overall compatibility, and an intermediate step uses an Airy stress function to ensure that the in-plane boundary conditions are properly upheld. The final set of governing equations follows from elementary calculus operations.

(*a*) *Strain energy densities*

The simplest approach considers the work done by the stresses, equation (2.5), on a deformed infinitesimal element of plate, first, when they are constant through the thickness, *t*, of plate and second, when they vary linearly across the depth but symmetrically about the middle surface, respectively, defining the strain energy densities in stretching, U_{S}, and in bending, U_{B}, for a general linear variation (Calladine 1983). For the first case

$$U_{\mathrm{S}} = \frac{t}{2}[\sigma_x\epsilon_x + \sigma_y\epsilon_y + \tau_{xy}\gamma_{xy}] = \frac{t}{2E}\left[\sigma_x^2 + \frac{\sigma_y^2}{\beta} - \frac{2\nu\sigma_x\sigma_y}{\beta} + \frac{\tau_{xy}^2}{\rho}\right], \tag{2.6}$$

after substituting for equation (2.4). The second case is handled more efficiently by introducing bending stress resultants, *M*, such that

$$M_x = \int_{-t/2}^{t/2}\sigma_x z\,\mathrm{d}z; \quad M_y = \int_{-t/2}^{t/2}\sigma_y z\,\mathrm{d}z; \quad M_{xy} = \int_{-t/2}^{t/2}\tau_{xy} z\,\mathrm{d}z. \tag{2.7}$$

The corresponding changes in curvature of the middle surface are χ_x, χ_y and χ_{xy}, the latter being the *engineering* twisting curvature, or simply 'twist', defined strictly within

$$\epsilon_x = z\chi_x; \quad \epsilon_y = z\chi_y; \quad \gamma_{xy} = 2z\chi_{xy}, \tag{2.8}$$

under Kirchhoff's hypothesis. Inserting the above into equation (2.5), and back into equation (2.7), and performing the integration across the thickness in the normal direction, *z*, the set of generalized Hooke's laws for orthotropic bending is obtained as

$$M_x = \frac{Et^3}{12(1-\nu^2/\beta)}[\chi_x + \nu\chi_y]; \quad M_y = \frac{Et^3\beta}{12(1-\nu^2/\beta)}\left[\chi_y + \frac{\nu\chi_x}{\beta}\right]; \quad M_{xy} = \frac{Et^3\rho}{6}\chi_{xy}. \tag{2.9}$$

Each of these moments performs work as the faces of the deforming element of plate rotate, and U_{B} can be calculated precisely from

$$\begin{aligned} U_{\mathrm{B}} &= \frac{1}{2}[M_x\chi_x + M_y\chi_y + 2M_{xy}\chi_{xy}] \\ &= \frac{Et^3}{24(1-\nu^2/\beta)}\left[\chi_x^2 + \beta\chi_y^2 + 2\nu\chi_x\chi_y + 4(1-\nu^2/\beta)\rho\chi_{xy}^2\right], \end{aligned} \tag{2.10}$$

when M_x, M_y and M_{xy} are substituted from equation (2.9).

(*b*) *Overall compatibility and Airy stress function*

Calladine (1983) gives the relationship between the change in Gaussian curvature, Δg, of the plate and the in-plane strains as

$$-\Delta g = \frac{\partial^2 \epsilon_y}{\partial x^2} + \frac{\partial^2 \epsilon_x}{\partial y^2} - \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}, \tag{2.11}$$

with Δg equal to $\kappa_x \kappa_y - \kappa_{xy}^2 - \kappa_{x0}\kappa_{y0} + \kappa_{xy0}^2$ by definition, where the final curvatures are κ_x, κ_y and κ_{xy}, and the initial curvatures are the same, but denoted by an extra subscript of '0'.

According to Mansfield (1989), the equilibrium condition for an element of plate without body forces can be expressed in terms of an Airy stress function, Φ, such that

$$\sigma_x = \frac{\partial^2 \Phi}{\partial y^2}; \quad \sigma_y = \frac{\partial^2 \Phi}{\partial x^2}; \quad \tau_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y}. \tag{2.12}$$

Substituting into equation (2.4), the strains in equation (2.11) are replaced by Φ, which can be differentiated and tidied, giving way to an amended compatibility statement

$$-E\Delta g = \frac{1}{\beta}\frac{\partial^4 \Phi}{\partial x^4} - \frac{\partial^4 \Phi}{\partial x^2 \partial y^2}\left[\frac{2\nu}{\beta} - \frac{1}{\rho}\right] + \frac{\partial^4 \Phi}{\partial y^4}. \tag{2.13}$$

(*c*) *Governing equations of deformation*

Solutions for Φ can be determined uniquely for the elliptical planform, when the major and minor semi-axes, of lengths a and b, are parallel to the same orthogonal coordinates, x and y. The initial shape of the shell is described by the set of distortions, w_0, of the middle surface normal to the xy plane, where $-w_0 = \kappa_{x0}x^2/2 + \kappa_{y0}y^2/2 + \kappa_{xy0}xy$. After deformation, the current transverse displacements are $-w = \kappa_x x^2/2 + \kappa_y y^2/2 + \kappa_{xy}xy$, and the quadratic variations of w and w_0 ensure that the changes in curvature are independent of x and y. The change in Gaussian curvature is also constant and hence equation (2.13) is satisfied by selecting a general fourth-order polynomial for Φ. Following up, the stresses vary quadratically from equation (2.12), and it can be verified that the following expressions

$$\sigma_x = S\left[x^2 + \frac{3a^2}{b^2}y^2 - a^2\right]; \quad \sigma_y = S\left[\frac{3b^2}{a^2}x^2 + y^2 - b^2\right]; \quad \tau_{xy} = -2Sxy, \tag{2.14}$$

uphold the boundary condition of zero force and shear force everywhere normal and tangential to the free edge while ensuring a symmetry of solution about the x- and y-axes. After substituting into the compatibility condition, equation (2.13), the constant, S, is calculated to be

$$S = -\frac{Ea^2b^2\Delta g}{\psi}, \tag{2.15}$$

with the dummy variable

$$\psi = 6\left[a^4 + \frac{b^4}{\beta} + \frac{a^2b^2}{3}\left(\frac{1}{\rho} - \frac{2\nu}{\beta}\right)\right]. \tag{2.16}$$

The information for the strain energy densities is now complete, enabling their integration over the elliptical planform to yield the total strain energy, U. A convenient dimensionless form, $\bar{U}$, derives from the groupings, $\bar{U}=U\cdot 12(1-\nu^2/\beta)a^3/E\pi t^5 b$, $\bar{\kappa}=\kappa\cdot a^2/t$ and $\phi=(1-\nu^2/\beta)b^4/\psi$, resulting in

$$\bar{U}=\frac{\phi}{2}\left[\bar{\kappa}_x\bar{\kappa}_y-\bar{\kappa}_{xy}^2-\bar{\kappa}_{x0}\bar{\kappa}_{y0}+\bar{\kappa}_{xy0}^2\right]^2+\frac{1}{2}\left[(\bar{\kappa}_x-\bar{\kappa}_{x0})^2+\beta(\bar{\kappa}_y-\bar{\kappa}_{y0})^2\right.$$
$$\left.+2\nu(\bar{\kappa}_x-\bar{\kappa}_{x0})(\bar{\kappa}_y-\bar{\kappa}_{y0})+4(1-\nu^2/\beta)\rho(\bar{\kappa}_{xy}-\bar{\kappa}_{xy0})^2\right]. \quad (2.17)$$

In the absence of externally applied loads, there are no other terms. Deformed, statical equilibrium configurations are therefore given by differentiating $\bar{U}$ with respect to $\bar{\kappa}_x$, $\bar{\kappa}_y$ and $\bar{\kappa}_{xy}$, and setting equal to zero, to reveal

$$\bar{\kappa}_x+\bar{\kappa}_y\mu-\bar{\kappa}_{x0}-\nu\bar{\kappa}_{y0}=0;\quad \beta\bar{\kappa}_y+\bar{\kappa}_x\mu-\beta\bar{\kappa}_{y0}-\nu\bar{\kappa}_{x0}=0;$$
$$(\alpha-\mu+\nu)\bar{\kappa}_{xy}-\alpha\bar{\kappa}_{xy0}=0, \quad (2.18)$$

where the extra terms, α and μ, have been introduced for compactness, such that

$$\alpha=2\rho(1-\nu^2/\beta), \quad (2.19)$$

and

$$\mu=\nu+\phi\left[\bar{\kappa}_x\bar{\kappa}_y-\bar{\kappa}_{xy}^2-\bar{\kappa}_{x0}\bar{\kappa}_{y0}+\bar{\kappa}_{xy0}^2\right]. \quad (2.20)$$

Note that α physically refers to the *torsional rigidity* of the shell with respect to the x- and y-axes; it is a dimensionless factor, which multiplies the flexural rigidity, $Et^3/12(1-\nu^2/\beta)$, in the x-direction. Equations (2.18) and (2.20) are four equations in the unknowns $\bar{\kappa}_x$, $\bar{\kappa}_y$, $\bar{\kappa}_{xy}$ and μ. By solving each of the curvatures explicitly,

$$\bar{\kappa}_x=\frac{\bar{\kappa}_{x0}(\beta-\mu\nu)+\bar{\kappa}_{y0}\beta(\nu-\mu)}{\beta-\mu^2};\quad \bar{\kappa}_y=\frac{\bar{\kappa}_{x0}(\nu-\mu)+\bar{\kappa}_{y0}(\beta-\mu\nu)}{\beta-\mu^2};\quad \bar{\kappa}_{xy}=\frac{\alpha\bar{\kappa}_{xy0}}{\alpha-\mu+\nu}. \quad (2.21)$$

The values of μ are obtained by substituting the above expressions back into equation (2.20), and solving the roots of the corresponding characteristic equation

$$(\nu-\mu)\left\{\phi\left[(\alpha-\mu+\nu)^2\left\{(\beta-\mu\nu)\left(\bar{\kappa}_{x0}^2+\beta\bar{\kappa}_{y0}^2\right)+\bar{\kappa}_{x0}\bar{\kappa}_{y0}(\beta\nu-3\mu\beta+\mu^2\nu+\mu^3)\right\}\right.\right.$$
$$\left.\left.+\bar{\kappa}_{xy0}^2(\beta-\mu^2)^2(2\alpha-\mu+\nu)\right]+(\beta-\mu^2)^2(\alpha-\mu+\nu)^2\right\}=0, \quad (2.22)$$

in terms of known values of $\bar{\kappa}_{x0}$, $\bar{\kappa}_{y0}$, $\bar{\kappa}_{xy0}$, β and ρ (and ϕ and α). The outside factor, $\nu-\mu$, simply confirms the initial state, $\bar{\kappa}_x=\bar{\kappa}_{x0}$, etc. but the remaining six roots present deformed equilibrium states. The stability of solutions can be assessed by confirming the positive definiteness of the generalized stiffness matrix

$$\begin{bmatrix} \partial^2 \bar{U}/\partial\bar{\kappa}_x^2 & \partial^2 \bar{U}/\partial\bar{\kappa}_x\partial\bar{\kappa}_y & \partial^2 \bar{U}/\partial\bar{\kappa}_x\partial\bar{\kappa}_{xy} \\ \partial^2 \bar{U}/\partial\bar{\kappa}_y\partial\bar{\kappa}_x & \partial^2 \bar{U}/\partial\bar{\kappa}_y^2 & \partial^2 \bar{U}/\partial\bar{\kappa}_y\partial\bar{\kappa}_{xy} \\ \partial^2 \bar{U}/\partial\bar{\kappa}_{xy}\partial\bar{\kappa}_x & \partial^2 \bar{U}/\partial\bar{\kappa}_{xy}\partial\bar{\kappa}_y & \partial^2 \bar{U}/\partial\bar{\kappa}_{xy}^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1+\phi\bar{\kappa}_y^2 & \mu+\phi\bar{\kappa}_y\bar{\kappa}_x & -2\phi\bar{\kappa}_y\bar{\kappa}_{xy} \\ \mu+\phi\bar{\kappa}_x\bar{\kappa}_y & \beta+\phi\bar{\kappa}_x^2 & -2\phi\bar{\kappa}_x\bar{\kappa}_{xy} \\ -2\phi\bar{\kappa}_{xy}\bar{\kappa}_y & -2\phi\bar{\kappa}_{xy}\bar{\kappa}_x & 2\alpha-2\phi\Delta\bar{g}+4\phi\bar{\kappa}_{xy}^2 \end{bmatrix}, \tag{2.23}$$

where the elements in the right matrix follow by further differentiating $\bar{U}$. Note that $\Delta\bar{g}$ is the change in dimensionless Gaussian curvature using $\bar{\kappa}$ terms.

This matrix follows from the second-order term in the Taylor expansion of the generalized energy expression (Guest & Pellegrino 2006), and is real and symmetrical. One condition for positive definiteness requires that all its eigenvalues are positive and real, and can be computed straightforwardly using a software package such as Matlab (MathWorks 2000).

3. Bistable configurations

The roots of the characteristic equation (2.22) can be extracted in closed form for specific initial conditions, most easily for isotropic behaviour. The results are similar to those found by Mansfield (1965), who deals with plates of tapering thickness, but one difference is the value taken by ϕ, which depends on the planform and thickness variation of the shell. For orthotropic behaviour, the roots do not oblige so easily and call for a numerical solution. Invariably, some of the roots are not real and physically impossible. There are also repeated roots, which detract further from the total number of stable shapes. In all the cases tested, it seems that only bistability is accorded, where one of the deformed configurations is stable. Nonetheless, given the assumption of uniform deformed curvatures, the corresponding bistable shape of structure is straightforward; and for cases that require a numerical solution, it is more instructive and visually simpler to present solution information concomitant to the range of initial shapes.

(*a*) *Isotropic behaviour*

Isotropic material behaviour is defined by setting $\beta=1$ and $\rho=1/2(1+\nu)$, returning $\alpha=1-\nu$, and the deformed curvatures in equation (2.21) are marginally simplified to

$$\bar{\kappa}_x = \frac{\bar{\kappa}_{x0}(1-\mu\nu)+\bar{\kappa}_{y0}(\nu-\mu)}{1-\mu^2}; \quad \bar{\kappa}_y = \frac{\bar{\kappa}_{x0}(\nu-\mu)+\bar{\kappa}_{y0}(1-\mu\nu)}{1-\mu^2}; \quad \bar{\kappa}_{xy} = \frac{(1-\nu)\bar{\kappa}_{xy0}}{1-\mu}, \tag{3.1}$$

and ignoring the initializing factor $\nu-\mu$, the characteristic equation, equation (2.22), becomes

$$\{\phi[(\bar{\kappa}_{x0}+\bar{\kappa}_{y0})^2(1-\mu\nu)+(\bar{\kappa}_{x0}\bar{\kappa}_{y0}-\bar{\kappa}_{xy0}^2)[\nu-2+\mu(2\nu-3)+\nu\mu^2+\mu^3]] \\ +(1-\mu^2)^2\}(1-\mu)^2=0. \tag{3.2}$$

The curvatures are undefined for the repeated roots, $\mu=1$, and the behaviour is governed by solutions to the fourth-order polynomial inside curly brackets.

(i) *Zero twist behaviour*

For the initially untwisted case, the polynomial is factorized more simply when $\bar{\kappa}_{x0}=\bar{\kappa}_{y0}=k_1$ or $\bar{\kappa}_{x0}=-\bar{\kappa}_{y0}=k_2$ for shells with equal (spherical) and opposite (saddle-shaped) principal curvatures, respectively

$$\{\phi k_1^2(2+\nu+\mu)+(1+\mu)^2\}(1-\mu)^2=0; \\ \{\phi k_2^2(2-\nu-\mu)+(1-\mu)^2\}(1+\mu)^2=0. \tag{3.3}$$

The unity factors do not define solutions, but the roots of the remaining quadratics ultimately yield a pair of real solutions conforming to inversions of the initial shapes, with new principal curvatures equal to

$$\bar{\kappa}_x=\bar{\kappa}_y=-\frac{k_1}{2}\left[1\pm[1-4(1+\nu)/\phi k_1^2]^{0.5}\right]; \\ \bar{\kappa}_x=-\bar{\kappa}_y=-\frac{k_2}{2}\left[1\pm[1-4(1-\nu)/\phi k_2^2]^{0.5}\right], \tag{3.4}$$

provided the constants k_1 and k_2 satisfy

$$k_1^2>4(1+\nu)/\phi; \quad k_2^2>4(1-\nu)/\phi. \tag{3.5}$$

The symmetry between the solutions is fortuitous, but the stability performances are somewhat different. Since there is no deformed twist from setting $\bar{\kappa}_{xy0}=0$, the generalized stiffness matrix from equation (2.23) can be reduced to

$$\begin{bmatrix} A & B & 0 \\ B & A & 0 \\ 0 & 0 & C \end{bmatrix} \Rightarrow \lambda_1=C, \quad \lambda_2=A+B, \quad \lambda_3=A-B, \tag{3.6}$$

where λ_1, λ_2 and λ_3 are the eigenvalues with $A=1+\phi\bar{\kappa}_y^2$, $B=\mu+\phi\bar{\kappa}_y\bar{\kappa}_x$ and $C=2\alpha-2\phi\Delta\bar{g}$. If the dummy parameters, η and ξ, are introduced such that

$$\eta=\frac{\phi k_1^2}{4(1+\nu)}, \quad \frac{\phi k_2^2}{4(1-\nu)}; \quad \xi=1\pm\sqrt{1-\frac{1}{\eta}}, \tag{3.7}$$

then real solutions are defined by $\eta\geq1$ and the eigenvalues can be written succinctly (see table 2).

In both cases, the solutions $0<\xi<1$ proffer negative eigenvalues irrespective of η, and the stiffness matrix is not positive definite, suggesting that the equilibria are unstable. When $1<\xi<2$, the magnitudes of the deformed curvatures are the

Table 2. Variation of the eigenvalues of stiffness matrix, equation (2.23), for the indicated initial shapes, $\bar{\kappa}_{x0}$ and $\bar{\kappa}_{y0}$, under isotropic conditions. There is no initial twist, and η and ξ are defined within equation (3.7). For the larger value of ξ, the sign of the eigenvalues is indicated inside parentheses.

eigenvalue	$\bar{\kappa}_{x0}=\bar{\kappa}_{y0}=k_1$	$\bar{\kappa}_{x0}=-\bar{\kappa}_{y0}=k_2$
$\lambda_1=C$	$2[1-\nu-\eta(1+\nu)(\xi^2-4)]$ (>0)	$2(1-\nu)[1+\eta(\xi^2-4)]$ (<0)
$\lambda_2=A+B$	$(1+\nu)[1+\eta(3\xi^2-4)]$ (>0)	$1+\nu+\eta(1-\nu)(4-\xi^2)$ (>0)
$\lambda_3=A-B$	$1-\nu-\eta(1+\nu)(\xi^2-4)$ (>0)	$(1-\nu)[1+\eta(3\xi^2-4)]$ (>0)

largest, and the eigenvalue performances are tabulated inside brackets in table 2. The equally curved case has a positive definite stiffness matrix, asserting bistability; however, for the oppositely curved shell, there is a single negative eigenvalue and hence no bistability. A separate study has confirmed that the latter case is, in fact, *metastable*, as the strain energy contours describe a saddle-like landscape around the equilibrium solution. In particular, the solution is unstable in the direction of twist, but stable for perturbations in either $\bar{\kappa}_x$ or $\bar{\kappa}_y$. Interestingly, Mansfield (1965) suggests bistability for the latter, but without formal substantiation.

The difference in stability is underpinned by the physical interpretation of the responsible eigenvalue, λ_1. From equation (2.23), λ_1 is equal to $\partial^2 \bar{U}/\partial\bar{\kappa}_{xy}^2=2\alpha-2\phi\Delta\bar{g}$, when there is no deformed twist. This expression is akin to the dimensionless torsional rigidity of the shell. Recall from equation (2.19) that α is the linear, or small-deflexion, component and $\phi\Delta\bar{g}$ accounts for the influence of large deflexions. Using the solutions from equation (3.4), the change in Gaussian curvature is positive and sufficiently large to render the oppositely curved case with negative torsional stiffness overall, and any twisting perturbation of the second equilibrium position simply destabilizes the shape, restoring the deflexions to zero. In the equally curved case, $\Delta\bar{g}$ is negative, but it contributes positively to the torsional stiffness. Insight is also gleaned from the in-plane stresses via equation (2.14), where the radial pattern of direct circumferential stress varies from compressive in the centre to tensile on the free edge. Thus, a hoop-wise tension forms over the bulk of the shell, which helps to reinforce the deformed shape against reversion. For the oppositely curved case, there is a circumferential band of compression that presumably augments instability, but it is less straightforward to visualize.

When $\bar{\kappa}_{x0}$ and $\bar{\kappa}_{y0}$ are generally different from each other, equation (3.2) has no obvious factors and numerical techniques are needed.

(ii) *Zero initial Gaussian curvature*

Setting $\bar{\kappa}_{x0}\bar{\kappa}_{y0}-\bar{\kappa}_{xy0}^2=0$, the fourth-order characteristic polynomial, equation (3.2), becomes

$$\phi(\bar{\kappa}_{x0}+\bar{\kappa}_{y0})^2(1-\mu\nu)+(1-\mu^2)^2=0, \tag{3.8}$$

whose roots can be determined in closed from, but not compactly for insight into stability. Instead, another approach advocated by Mansfield considers the magnitude of the polynomial's discriminant, which depends on the product of the differences between the roots. When the discriminant is negative, some of the roots are certainly real, indicating possible bistable behaviour. Although determining the roots or discriminant of low-order polynomials poses few numerical problems, the latter has a more succinct closed form. Writing the above equation as

$$\mu^4 + a_2\mu^2 + a_3\mu + a_4 = 0; \quad a_2 = -2, \quad a_3 = -\nu\phi(\bar{\kappa}_{x0} + \bar{\kappa}_{y0})^2,$$
$$a_4 = 1 + \phi(\bar{\kappa}_{x0} + \bar{\kappa}_{y0})^2, \tag{3.9}$$

its discriminant is equal to Weisstein (2006)

$$-4a_3^2a_2^3 - 27a_3^4 + 256a_4^3 + a_4(16a_2^4 + 144a_3^2a_2) - 128a_4^2a_2^2, \tag{3.10}$$

and specifically here to

$$256\phi^2(\bar{\kappa}_{x0} + \bar{\kappa}_{y0})^4\left[1 - \nu^2 + \phi(\bar{\kappa}_{x0} + \bar{\kappa}_{y0})^2\left[1 - \frac{9\nu^2}{8}\right] - \phi^2(\bar{\kappa}_{x0} + \bar{\kappa}_{y0})^4\frac{27\nu^4}{256}\right]. \tag{3.11}$$

When ν takes values around 1/3 for engineering materials, it may be verified that the discriminant is positive provided $\phi(\bar{\kappa}_{x0} + \bar{\kappa}_{y0})^2 < 673$, which, in reality, is not upheld only for highly curved strips. The behaviour of shallow strips complies with a positive discriminant, and the roots of equation (3.8) are imaginary resulting in no bistability. This result applies equally to singly curved strips, which have no initial Gaussian curvature, and they too are never bistable. Guest & Pellegrino (2006) also establish this result under their strict assumption of inextensional behaviour.

(iii) *General initial distortions*

For any initial shape, the previous discriminant approach can be employed, but it does not quantify the stability of shape. Algorithmically, it can cause difficulties in view of the real behaviour. Repeated factors in the characteristic equation register zero discriminant, even though real roots and possibly bistable solutions persist. Therefore, the roots of equation (3.2) are solved numerically, the deformed curvatures are computed, and the elements of the stiffness matrix in equation (2.23) are determined for assessing its positive definiteness. For example, figure 1 shows that for zero initial twist, bistable configurations exist, provided the combination of $\bar{\kappa}_{x0}$ and $\bar{\kappa}_{y0}$ values is large enough and of the same sign. The earlier closed-form predictions, §3*a*(i), confirm solutions along the lines $\bar{\kappa}_{y0} = \pm\bar{\kappa}_{x0}$. The performances under increasing aspect ratio are resolved over the sub-figures up to a value of $b/a = 100$, which approximates the response of a thin, but doubly curved narrow strip. In all cases, there is little difference in the overall form, although the bistable regions expand asymptotically towards the origin and symmetrically about the line $\bar{\kappa}_{x0} = \bar{\kappa}_{y0}$.

Formal solutions for the effects of initial twist can also be computed, but are not needed in view of the following argument. Using a Mohr's circle of twist versus curvature (Calladine 1983) and equations (3.1), the following invariant relationship can be proven

$$\frac{\bar{\kappa}_x - \bar{\kappa}_y}{\bar{\kappa}_{xy}} = \frac{\bar{\kappa}_{x0} - \bar{\kappa}_{y0}}{\bar{\kappa}_{xy0}}, \tag{3.12}$$

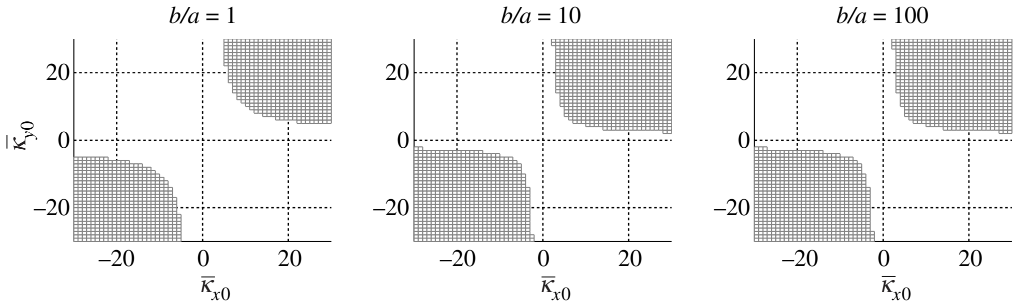

Figure 1. Regions of feasible bistable geometries are shaded grey for isotropic (ν=0.3) elliptical shells under increasing aspect ratio, b/a, with $\bar{\kappa}_{xy0}=0$. Unshaded regions are configurations which are not bistable.

which is also confirmed by Mansfield (1965). This result asserts that the axes of principal curvature, before and after deformation, are coincident. Any case with initial twist can be transformed to principal axes, but crucially, the initial values of average curvature and Gaussian curvature are the same as $\bar{\kappa}_{x0}+\bar{\kappa}_{y0}$ and $\bar{\kappa}_{x0}\bar{\kappa}_{y0}-\bar{\kappa}_{xy0}^2$, so that the roots in equation (3.2) are unaffected by the transformation. Thus, the results of cases without twist—as in figure 1—can be transcribed directly to cases, which include twist. Moreover, for shells with negative initial Gaussian curvature, adding twist simply retains principal curvatures of opposite senses, and no improvement towards bistability is obtained. For positive Gaussian curvature, the principal curvatures also diverge upon adding twist, which may result in a change in sign of the Gaussian curvature, and the effects of twist, generally speaking, diminish bistable performance. Correspondingly, the bistable boundaries in figure 1 move progressively further from the origin under increasing initial twist.

Isotropy guarantees the invariance of behaviour imbued in equation (3.12), but for orthotropic behaviour, all cases involving initial twist must be reviewed separately.

(*b*) *Orthotropic behaviour*

It is worth commenting upon the deformed curvatures in equation (2.21) without a formal solution. A singly curved shell, where $\bar{\kappa}_{x0}=0$ or $\bar{\kappa}_{y0}=0$, has a second, orthogonally curved configuration only if μ takes the value β/ν, resulting in the non-zero curvature, $\nu\times(\bar{\kappa}_{x0},\bar{\kappa}_{y0})$. However, this prospective solution is not a viable root of the characteristic polynomial, equation (2.22), and a strip without initial Gaussian curvature must stretch for bistability, where the new shape is doubly curved. A second comment is that when $\mu=\alpha+\nu$, then the deformed twist, $\bar{\kappa}_{xy}$, is not defined by its expression inside equation (2.21), rather, it must be calculated from μ via equation (2.20). This is relevant, when $\bar{\kappa}_{xy0}$ is zero; the characteristic equation has a repeated factor, $(\alpha+\nu-\mu)^2$, and the values of $\bar{\kappa}_{x0}$ and $\bar{\kappa}_{y0}$ determine whether $\bar{\kappa}_{xy}$ will be real.

(i) *Direct stress 'isotropy'*

Closed-form solutions are available, when β is set equal to unity. Assuming material homogeneity, this condition defines isotropic behaviour for direct stresses and, consequently, no directional dependence of the flexural rigidities,

Table 3. Variation of the eigenvalues of stiffness matrix, equation (2.23), for the indicated initial shapes under semi-orthotropic behaviour ($\beta=1$, but $\alpha\neq 1-\nu$). η and ξ are defined within equation (3.14).

eigenvalue	$\bar{\kappa}_{xy0}=k_3,\ \bar{\kappa}_{x0}=\bar{\kappa}_{y0}=0$	$\bar{\kappa}_{x0}=-\bar{\kappa}_{y0}=k_4,\ \bar{\kappa}_{xy0}=0$
λ_1	$2\alpha[1-\eta(4-3\xi^2)]$	$2\alpha-2(1-\nu)\eta(4-\xi^2)$
λ_2	$1+\nu+\alpha\eta(4-\xi^2)$	$1+\nu+\eta(1-\nu)(4-\xi^2)$
λ_3	$(1-\nu)-\alpha\eta(4-\xi^2)$	$(1-\nu)[1-\eta(4-3\xi^2)]$

but it allows freedom to proportion the effect of independent shear modulus, ρE, and the torsional rigidity of the shell. The two cases are defined by a state of pure twist, $\bar{\kappa}_{xy0}=k_3$ and $\bar{\kappa}_{x0}=\bar{\kappa}_{y0}=0$, and by equal and opposite principal curvatures, $\bar{\kappa}_{x0}=-\bar{\kappa}_{y0}=k_4$ and $\bar{\kappa}_{xy0}=0$. The characteristic polynomial, equation (2.22), admits only soluble quadratic factors

$$\phi k_3^2(2\alpha+\nu-\mu)+(\alpha+\nu-\mu)^2=0\Rightarrow\mu=\alpha+\nu+\frac{\phi k_3^2}{2}\left[1\pm\left[1-4\alpha/\phi k_3^2\right]^{0.5}\right];$$

$$\phi k_4^2(2-\nu-\mu)+(1-\mu)^2=0\Rightarrow\mu=1+\frac{\phi k_4^2}{2}\left[1\pm\left[1-4(1-\nu)/\phi k_4^2\right]^{0.5}\right]. \quad (3.13)$$

The deformed curvatures are determined from equation (2.21), and the eigenvalues, λ_1, λ_2 and λ_3, of the stiffness matrix, equation (2.23), can be written in closed form using the dummy variables

$$\eta=\frac{\phi k_3^2}{4\alpha},\quad \frac{\phi k_4^2}{4(1-\nu)};\quad \xi=1\pm\sqrt{1-\frac{1}{\eta}}, \quad (3.14)$$

where $\eta\geq 1$ accords real solutions (see table 3).

When $\xi=1-\sqrt{1-1/\eta}$, the eigenvalue signs in both cases are not consistent, and these cannot be bistable solutions. For the case of pure twist, λ_1 and λ_2 are always positive, when ξ adopts the larger value, and $\lambda_3>0$ is assured, when $\alpha<(1-\nu)/3$ for $\eta\geq 1$. Similarly, for the oppositely curved case, $\lambda_2>0$ and $\lambda_3>0$, but $\lambda_1>0$ only when $\alpha>3(1-\nu)$. Either case can be bistable depending on the value of α, but not together, and the deformed curvatures, without explicitly stating them, give rise to an inversion of initial shape. In the isotropic case, α cannot deviate from $1-\nu$, which lies outside both inequalities, and the unstable regions for oppositely curved cases in figure 1 are again confirmed.

For the oppositely curved case, setting $\alpha>3(1-\nu)$ for bistability has a direct physical understanding based on the discussion of isotropy in §3*a*(i). The eigenvalue, λ_1, here also equates to the large deflexion torsional stiffness, and the linear component, α, is sufficiently large to nullify any reduction in the overall rigidity foisted by a positive change in the Gaussian curvature during deformation. However, a seemingly paradoxical feature stems from regarding the twisted but bistable case ($\alpha<(1-\nu)/3$) as an initial shape with equal and opposite principal curvatures along the directions defined by $y=\pm x$. As a corollary to this case, the orientation of a fixed shape without axisymmetry with respect to the original orthotropic axes determines whether there can be bistability. The issue is, however, simply resolved, when the equivalent orthotropic properties associated with the orientation of shape are discerned. For example, if the second coordinate axes, X and Y, are aligned to $y=\pm x$, then separate Mohr's circles of stress and strain admit the following relationships

$$\sigma_x = \frac{\sigma_X}{2} + \frac{\sigma_Y}{2} - \tau_{XY}; \quad \sigma_y = \frac{\sigma_X}{2} + \frac{\sigma_Y}{2} + \tau_{XY}; \quad \tau_{xy} = \frac{\sigma_X - \sigma_Y}{2}, \tag{3.15}$$

$$\epsilon_X = \frac{\epsilon_x}{2} + \frac{\epsilon_y}{2} + \frac{\gamma_{xy}}{2}; \quad \epsilon_Y = \frac{\epsilon_x}{2} + \frac{\epsilon_y}{2} - \frac{\gamma_{xy}}{2}; \quad \gamma_{XY} = \epsilon_x - \epsilon_y,$$

and the equivalent constitutive relationships are forged using equation (2.4), where β is set to unity. The Young's moduli are $E_X = E_Y = \bar{E}$, the Poisson's ratios are $\nu_{YX} = \nu_{XY} = \bar{\nu}$ and the independent shear modulus is $\bar{\rho}\bar{E}$. After rearrangement, the terms are found to be

$$\bar{E} = \frac{2E}{1-\nu+(1+\nu)/\zeta}; \quad \bar{\nu} = \frac{(1+\nu)/\zeta - 1 + \nu}{(1+\nu)/\zeta + 1 - \nu}; \quad \bar{\zeta} = \frac{1}{\zeta} \quad \text{with}$$

$$\zeta = 2\rho(1+\nu), \quad \bar{\zeta} = 2\bar{\rho}(1+\bar{\nu}). \tag{3.16}$$

Importantly, ζ measures the performance of the modular ratio in shear, ρ, relative to its isotropic value of $1/2(1+\nu)$. The larger ζ becomes, the greater the value of α, which affords bistability for the original oppositely curved case, where $\bar{\kappa}_{x0} = \bar{\kappa}_{y0} \neq 0$ and $\bar{\kappa}_{xy0} = 0$. Correspondingly, $\bar{\zeta}$ defines the same performance for $\bar{\rho}$, but for parameters evaluated in the XY coordinate system. Owing to the reciprocal relationship with ζ, any augmentation of the original shear modulus reduces the equivalent XY component. If so, then the stability of a shell curved in opposite directions along $y=\pm x$ (and therefore twisted w.r.t. x- and y-axes) is governed by insufficient torsional rigidity, as it is less than the isotropic value, and there can be no bistability.

Although the above relates to a specific example, the general implications relate once again to the effects of adding twist to a principally curved shape, where again, it can diminish bistable properties. This feature is pertinent in the manufacture of composites, where asymmetry during curing and associated thermal creep effects can introduce unexpected twist distortions, which may detract from perceived bistability. These effects are demonstrated in §3*b*(ii) using a practical example.

(ii) *A numerical example*

Ranges of bistable solutions are now obtained, in which the general sixth-order characteristic polynomial, equation (2.22), is solved numerically excluding the initializing factor $\mu=\nu$. For brevity, the values of the relative constants are taken from a 'composite' material in Guest & Pellegrino (2006), which exhibits orthotropic bending with $\beta=0.977$, $\nu=0.766$, and $\rho=1.965$. Note that since β is almost equal to unity, the closed-form procedures in §3*b*(i) offer complementary insight. Of course, the results may not apply to composites that display anisotropy, for example, anti-symmetrical fibre lay-ups, which are coupled between twisting and stretching even though the uncoupled constitutive

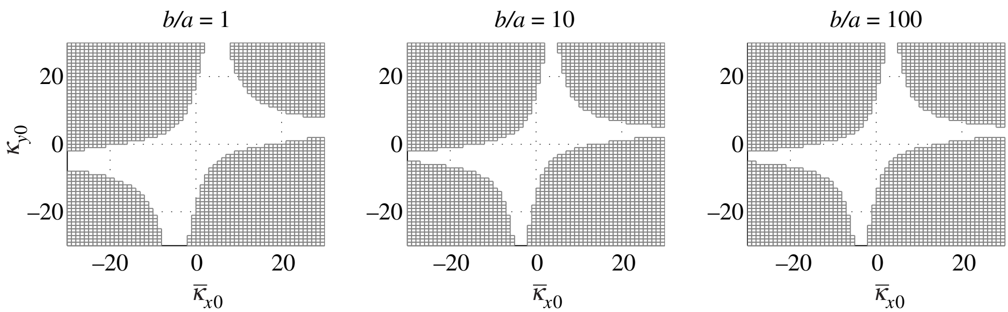

Figure 2. Regions of bistable configurations (shaded) for an orthotropic ($\beta=0.977$, $\nu=0.766$, $\rho=1.965$) elliptical shell of increasing aspect ratio, b/a, with no initial twist. Unshaded regions are not feasible solutions.

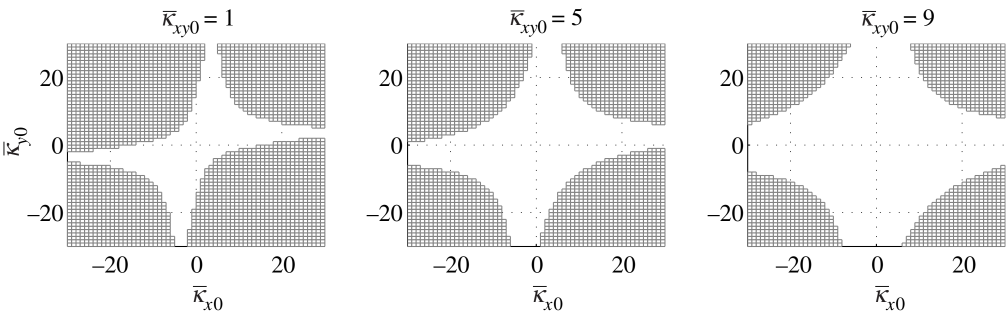

Figure 3. Regions of bistable configurations (shaded) for an orthotropic ($\beta=0.977$, $\nu=0.766$, $\rho=1.965$) elliptical shell ($b/a=2$) with increasing initial twist, $\bar{\kappa}_{xy0}$. Unshaded regions are not feasible solutions.

behaviour is orthotropic, and symmetrical lay-ups, which proffer flexural and twisting coupling out-of-plane alone. A formal study would be required to address their behaviour, but the immediate aim here is to distill new features pertaining to orthotropic behaviour alone.

For structures with zero initial twist and increasing aspect ratio, figure 2 shows that bistability prevails for initial configurations, where the Gaussian curvature is positive and negative, provided $\bar{\kappa}_{x0}$ and $\bar{\kappa}_{y0}$ are large enough in combination. All bistable solutions are orthogonally curved without twist. A significant feature is the fivefold increase in the modular ratio, ρ, which sufficiently elevates the torsional rigidity, α. Note also that the feasible bistable regions intersect both axes, $\bar{\kappa}_{x0}=0$ and $\bar{\kappa}_{y0}=0$, indicating that singly curved structures are bistable, when $|\bar{\kappa}_{x0}|$ or $|\bar{\kappa}_{y0}|$ is roughly greater than 15. As b/a increases, this threshold reduces but only marginally.

Supporting the discussion in §3b(i), figure 3 indicates that the action of adding initial twist detracts from the bistability of doubly curved cases. As $\bar{\kappa}_{xy0}$ increases, the levels of initial curvature for bistable behaviour must also increase. Commensurately, singly curved shells are no longer feasible configurations, when $\bar{\kappa}_{xy0}>5$ for the chosen range of $\bar{\kappa}_{x0}$ and $\bar{\kappa}_{y0}$.

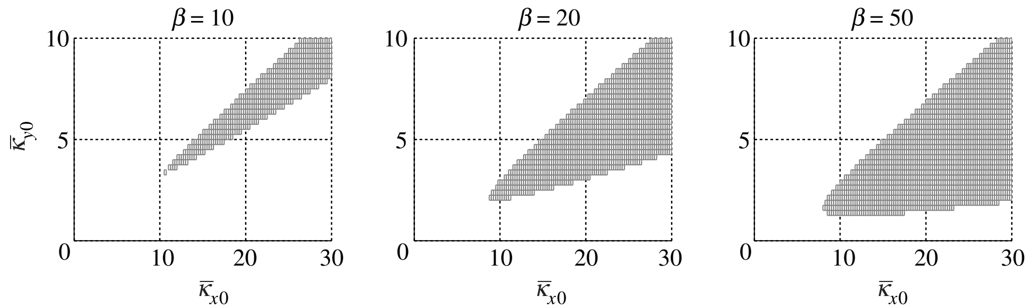

Figure 4. Regions of bistability (shaded) for an elliptical shell ($b/a=2$) with $\nu=0.3$ and $\rho=1/2(1+\nu)$ for increasing modular ratio, β. The initial Gaussian curvature is set to be zero everywhere, so that $\bar{\kappa}_{xy0}=\sqrt{\bar{\kappa}_{x0}\bar{\kappa}_{y0}}$. Unshaded regions are not feasible.

(iii) *A synthetic material solution*

As a final theoretical exercise, let us consider the case where the material properties can be proportioned systematically. In the following example, only the β term is varied, with ν and ρ taking isotropic values. This might be contrived in practice by fixing transverse narrow strips of material periodically across the surface of a thin metallic sheet. Furthermore, the initial Gaussian curvature is arbitrarily set to zero, and the range of bistable solutions is displayed in figure 4. For example, for the initial point, $(\bar{\kappa}_{x0},\bar{\kappa}_{y0})=(20,5)$, the bistable configuration when $\beta=50$ reveals $\bar{\kappa}_x=12.9$, $\bar{\kappa}_y=4.63$ and $\bar{\kappa}_{xy}=-7.41$, resulting in a small but moderate increase in Gaussian curvature. The initial structure is cylindrically curved along an axis between both semi-axes, and the new configuration is almost cylindrical, but aligned on the opposite side of the x-axis.

4. Conclusions

This study has obtained the governing equations of large deflexion behaviour for an elliptical shell with orthotropic material properties and uniform shape, in order to investigate bistable behaviour without loads and pre-stress and its dependent factors. The formulation is approximate, as it prescribes uniform changes in curvature everywhere in the shell, including regions near to the free edge, where a practical non-uniform boundary layer must develop. The assumption of uniformity can be justified, however, if the shell is very thin. Mansfield (1989) finds an approximate width of boundary layer equal to 0.77 $\sqrt{t/\kappa}$ for a flat strip bent by end couples to a curvature, κ. For the deformed stable curvatures, this width commutes of the order of a few thickness, although a more formal study is required. The assumption, nevertheless, leads to a compact set of governing equations for determining new equilibrium states, whose stability is assessed via the positive definitiveness of a generalized 3×3 stiffness matrix. In particular, this study has yielded the following novel results. For isotropic material behaviour, a shell is only bistable when it possesses sufficient positive Gaussian curvature. Other configurations are unstable or at least metastable. Mansfield (1965) obtains an exact solution of behaviour using a precise tapering *lenticular* cross-section, and suggests that oppositely curved shells are *possibly* bistable. The difference here is not due to the underlying assumptions, as the governing equations in equations (2.18) and (2.22) can be reduced to expressions identical to Mansfield, when β and ρ are set to isotropic values of 1 and $1/2(1+\nu)$. The make-up of the ϕ term in equation (2.17) is different, as it is essentially a shape factor; but it does not alter the character of the general stability criteria, rather, it defines different thresholds at which bistability is accorded. For orthotropic material, inextensional deformation would appear not to be a viable transition mode between equilibria. Guest & Pellegrino (2006) assume inextensional behaviour, as it best describes the practical response of their rectangular strips, but limits the range of possible geometrical configurations for study. Again, the difference here is due to the number of deformed 'output' parameters. There are three output parameters in $\bar{\kappa}_x$, $\bar{\kappa}_y$ and $\bar{\kappa}_{xy}$, and the assumption of inextensional behaviour would constrain them to each other by zero change in the Gaussian curvature. As a consequence of a non-zero change, bistability also depends on the magnitudes of initial curvatures, which govern the relative variation between the stretching and bending strain energy densities, even for cylindrical shells with no initial Gaussian curvature; neglecting extensional effects does not capture or reveal any such threshold concerning the magnitude of initial shape. It has also been shown that orthotropic shells can be bistable for configurations with negative as well as positive initial Gaussian curvature. The reason bonds with the stability criteria, where the likelihood of positive definiteness of the stiffness matrix is increased by having a larger torsional rigidity, an effect physically tantamount for increasing the material shear modulus. Careful attention must also be paid to the degree of initial twist, as it has been demonstrated that depending on the choice of twist axes, it can either complement or undermine the bistable capabilities of the shell. In terms of practicability, the shell is invariably connected or fixed along its boundary, thereby affecting the free-edge condition. Undoubtedly, its bistability is also affected, but the precise changes in performance demand further scrutiny. As a final comment, the original motivation for the synthesis exercise in §3*b*(iii) was to expose *multiply-stable* configurations, given that the characteristic polynomial in equation (2.22) can yield up to six possible roots. Even for a wide range of orthotropic properties, only a single viable root for a deformed shape was obtained, and further refinements to this aspect of study will require the inclusion of material anisotropy and, possibly, non-uniform shapes.

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